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The fluid-filled cylindrical membrane container

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SUMMARY
The static shape of a fluid-filled membrane cylinder can be described by a set of nonlinear differential equations. These equations depend on a nondimensional parameter β representing the relative importance between pressure and the gravity force. The solution is found by three methods: similarity solution for small β, asymptotic solution for large β, and numerical integration.

1. Introduction and formulation

Fabric membrane containers are desirable in the storage and transportation of liquids due to their low cost and their ease of set up and dismantability. Long cylindrical flexible containers, filled with sand or water, have also been used as dams in flood control [1, 2]. The present paper is a theoretical study of such fluid-filled membrane containers.

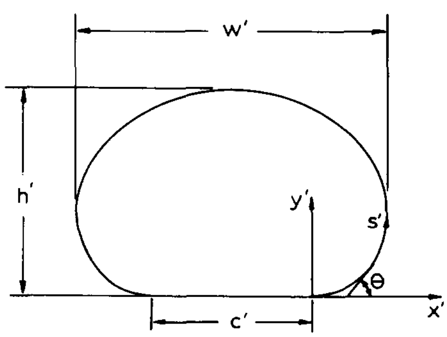

Figure 1. The coordinate axes.

Fig. 1 shows the cross section of a fluid-filled cylinder resting on the ground. We shall assume the membrane fabric is of negligible density, inextensible, and has a fixed total perimeter of length L. Fluid of density ρ can be pumped through a hole on the bottom with pressure p_0.

The local curvature at any point is directly proportional to the local pressure difference and inversely proportional to membrane tension. Using the coordinate axes shown, the equations governing the shape of the cylinder are

$$\text{curvature} = \frac{d\theta}{ds'} = \frac{p_0 - p_a - \rho g y'}{T}, \tag{1}$$

$$\frac{dx'}{ds'} = \cos\theta, \quad \frac{dy'}{ds'} = \sin\theta \tag{2}$$

where p_a is the ambient pressure, g is the gravitational acceleration, T is the tensile force experienced by the membrane, s' is the arc length from the origin, and θ is the local angle of inclination. We shall normalize all lengths by L and drop primes. The governing equations become

$$\frac{d\theta}{ds} = \frac{1}{\alpha}(\beta - y), \tag{3}$$

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta \tag{4}$$

where $\beta \equiv (p_0 - p_a)/\rho g L$ and $\alpha \equiv T/\rho g L^2$ are non-dimensional parameters. Since T is constant, the boundary conditions are

$$\text{at } s = 0: \quad x = y = \theta = 0, \tag{5}$$

$$\text{at } s = 1 - c: \quad x = -c, \quad y = 0, \quad \theta = 2\pi, \tag{6}$$

where c is the fractional length of perimeter which touches the ground. Given β, the unknowns are α, c, x, y, θ.

If we differentiate (3) with respect to s, multiply by $d\theta/ds$ and integrate twice, we obtain

$$s = \sqrt{\frac{\alpha}{2}} \int_0^\theta \frac{d\theta}{\sqrt{a + \cos\theta}} = \sqrt{\frac{2\alpha}{1+a}} \left[E\left(\sqrt{\frac{2}{a+1}}, \frac{\theta}{2}\right) - E\left(\sqrt{\frac{2}{a+1}}, 0\right) \right]. \tag{7}$$

Here a is an integration constant and E is the elliptic integral of the first kind. However, it is almost impossible to complete the solution analytically from (4)-(7), since the unknowns θ and a are implicit in the elliptic functions. In what follows we shall introduce several other methods in order to give more insight into the character of the solution.

2. The similarity solution when β is small

This is the case when the membrane cylinder contains very little liquid. Most of the cylinder surface has zero slope except at the sides where $s \approx 0$ and $s \approx 1/2$. We introduce a similar variable ζ,

$$\zeta \equiv s/\sqrt{\alpha}. \tag{8}$$

Eq (3), after differentiation, becomes

$$\frac{d^2\theta}{d\zeta^2} = -\sin\theta. \tag{9}$$

This can be integrated to yield

$$\frac{d\theta}{d\zeta} = \sqrt{2(1+\cos\theta)} \tag{10}$$

where the constant of integration has been determined by

$$\zeta \to \infty, \quad \theta \to \pi, \quad \frac{d\theta}{d\zeta} \to 0. \tag{11}$$

When $\theta = 0$, $y = 0$, Eqs (3), (10) give

$$\alpha = \beta^2/4 \ll 1. \tag{12}$$

The exact solution to (10), (4) is

$$\frac{s}{\sqrt{\alpha}} = \zeta = \ln\tan\left(\frac{\theta+\pi}{4}\right), \tag{13}$$

$$\frac{x}{\sqrt{\alpha}} = 2\sin\frac{\theta}{2} - \ln\tan\left(\frac{\theta+\pi}{4}\right), \tag{14}$$

$$\frac{y}{\sqrt{\alpha}} = 2 - 2\cos\frac{\theta}{2}. \tag{15}$$

We see that when β is small, the shape of the cylinder (14), (15) is similar, i.e. directly proportional to β (See Fig. 2). From (14),

$$x\Big|_{\theta=\pi} = -s + 2\sqrt{\alpha}\sin\frac{\theta}{2}\Big|_{\theta=\pi} = -s + 2\sqrt{\alpha}. \tag{16}$$

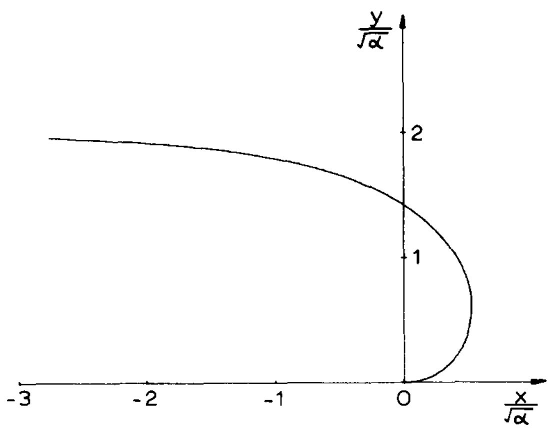

Figure 2. The similarity profile for small β.

We find

$$c = \frac{1}{2}\,[1 - 2(x\,\big|_{\theta=\pi} + s)] = \frac{1}{2} - 2\sqrt{\alpha} = \frac{1}{2} - \beta\,. \tag{17}$$

Let w and h denote the normalized width and height of the membrane tube. Then

$$w = c + 2x\,\Big|_{\theta=\frac{\pi}{2}} = \frac{1}{2} + (\sqrt{2} - 1 - \ln\tan\frac{3\pi}{8})\,\beta\,, \tag{18}$$

$$h = y\,\big|_{\theta=\pi} = \beta\,. \tag{19}$$

The volume per unit length of cylinder V' can be computed as follows:

$$V \equiv \frac{V'}{L^2} = \frac{(p_0 - p_a)c}{\rho g L} = \beta c = \beta/2 - \beta^2\,. \tag{20}$$

3. Asymptotic solution when β is large

Let $\beta \equiv 1/\epsilon$ where $\epsilon \ll 1$. This is the case when the membrane cylinder is so pressurized that the cross-sectional shape is almost a circle. Let us perturb about this limiting case by setting

$$\theta = 2\pi s + \epsilon\,\theta_1(s) + \epsilon^2\,\theta_2(s) + \ldots\,, \tag{21}$$

$$x = \frac{1}{2\pi}\sin 2\pi s + \epsilon\,x_1(s) + \epsilon^2 x_2(s) + \ldots\,, \tag{22}$$

$$y = \frac{1}{2\pi}(1 - \cos 2\pi s) + \epsilon\,y_1(s) + \epsilon^2 y_2(s) + \ldots\,, \tag{23}$$

$$\frac{\beta}{\alpha} = 2\pi + \epsilon A_1 + \epsilon^2 A_2 + \ldots\,, \tag{24}$$

$$c = 0 + \epsilon\,c_1 + \epsilon^2 c_2 + \ldots\,. \tag{25}$$

Here the gauge functions (ϵ^n) are determined uniquely from (3). Substitution of (21)-(25) into (3)-(5) yields the following successive equations:

$$\frac{d\theta_1}{ds} = A_1 + \cos 2\pi s - 1,\ \frac{dx_1}{ds} = -\theta_1\sin 2\pi s,\ \frac{dy_1}{ds} = \theta_1\cos 2\pi s\,, \tag{26}$$

$$\frac{d\theta_2}{ds} = A_2 + \frac{A_1}{2\pi}(\cos 2\pi s - 1) - 2\pi y_1\,, \tag{27}$$

$$\frac{dx_2}{ds} = \frac{-\theta_1^2}{2}\cos 2\pi s - \theta_2\sin 2\pi s\,,\ \frac{dy_2}{ds} = \frac{-\theta_1^2}{2}\sin 2\pi s + \theta_2\cos 2\pi s, \tag{28}$$

$$\theta_1(0) = x_1(0) = y_1(0) = 0\,,\quad \theta_2(0) = x_2(0) = y_2(0) = 0\,. \tag{29}$$

Eq. (6) is not so straight-forward. For instance, the condition on θ is

$$\begin{aligned}\theta\Big|_{s=1-c} &= 2\pi(1-c)+\epsilon\,\theta_1(1-c)+\epsilon^2\theta_2(1-c)+\ldots\\ &= 2\pi(1-\epsilon c_1-\epsilon^2 c_2)+\epsilon\,\theta_1(1)-\epsilon^2 c_1\frac{d\theta_1}{ds}(1)+\epsilon^2\theta_2(1)+\ldots\\ &= 2\pi.\end{aligned} \tag{30}$$

From (30), the boundary conditions are

$$\theta_1(1)=2\pi c_1,\quad x_1(1)=0,\quad y_1(1)=0, \tag{31}$$

$$\theta_2(1)=2\pi c_2+c_1\frac{d\theta_1}{ds}(1),\ x_2(1)=c_1\frac{dx_1}{ds}(1),\ y_2(1)=-c_1^2\pi+c_1\frac{dy_1}{ds}(1). \tag{32}$$

The solutions, after some algebra, are

$$A_1=\frac{3}{2},\quad c_1=\frac{1}{4\pi},\quad \theta_1=\frac{1}{2\pi}(\pi s+\sin 2\pi s), \tag{33}$$

$$x_1=\frac{-1}{16\pi^2}(2\sin 2\pi s-4\pi s\cos 2\pi s-4\pi s-\sin 4\pi s), \tag{34}$$

$$y_1=\frac{1}{16\pi^2}(2\cos 2\pi s+4\pi s\sin 2\pi s-1-\cos 4\pi s), \tag{35}$$

$$A_2=\frac{3}{4\pi},\quad c_2=0, \tag{36}$$

$$\theta_2=\frac{1}{8\pi^2}(\sin 2\pi s+2\pi s\cos 2\pi s+\frac{1}{4}\sin 4\pi s+\pi s), \tag{37}$$

$$x_2=\frac{1}{32\pi^3}[\sin^3 2\pi s-2\sin 2\pi s(\pi s+\sin 2\pi s)^2], \tag{38}$$

$$y_2=\frac{1}{32\pi^3}[\cos^3 2\pi s-1+2\cos 2\pi s(\pi s+\sin 2\pi s)^2]. \tag{39}$$

From (24), (25), (33), (36) we find

$$\alpha=\frac{\beta}{2\pi+\epsilon A_1+\epsilon^2 A_2}=\frac{\beta}{2\pi}(1-\frac{3}{4\pi\beta}+\frac{3}{16\pi^2\beta^2}+\ldots), \tag{40}$$

$$c=\frac{1}{4\pi\beta}+O(\beta^{-3}). \tag{41}$$

For the width and height we need the values of s at $\theta=\pi/2$ and π. For example, at $\theta=\pi/2$, the inversion of (21) is obtained by the expansion

$$s=\frac{1}{4}+\epsilon s_1+\epsilon^2 s_2+\ldots \tag{42}$$

and expressing θ_1, θ_2 in a Taylor series about 1/4. Then (42) is substituted into (22) for x at $\theta = \pi/2$. The width is

$$w = c + 2x \Big|_{\theta=\frac{\pi}{2}} = \frac{1}{\pi} + \frac{\pi-2}{8\pi^2\beta} + \frac{1}{16\pi^3\beta^2} + \ldots. \tag{43}$$

Similarly the height is

$$h = y \Big|_{\theta=\pi} = \frac{1}{\pi} - \frac{1}{4\pi^2\beta} - \frac{1}{16\pi^3\beta^2} + \ldots. \tag{44}$$

The normalized volume is

$$V = \beta c = \frac{1}{4\pi} + \ldots. \tag{45}$$

4. Numerical solution

With the advent of computers, the present problem is perhaps more easily solved exactly by direct numerical integration of (3)-(6) than numerically for the unknown constants through elliptic functions. The method is to guess α and integrate (3)-(5) as an initial-value problem using the Runge-Kutta algorithm. When $\theta = 2\pi$, we check whether $s = 1 + x$. If not, α is adjusted, say, by Newton's method. Convergence is not a problem here because our analytical solutions, (12)

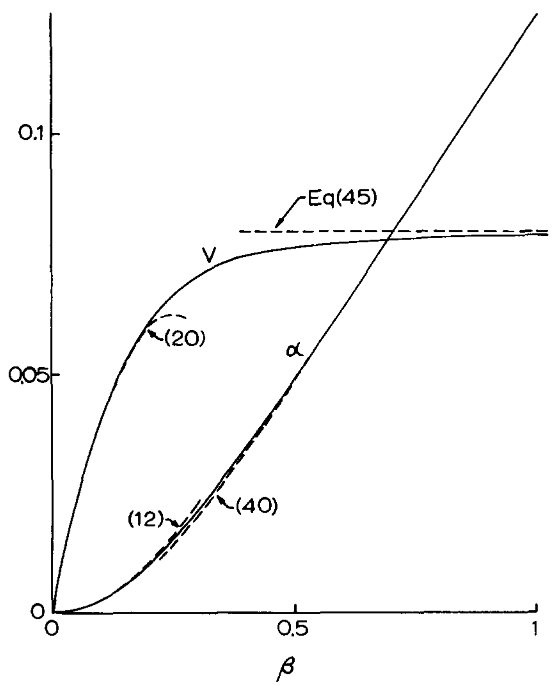

Figure 3. Volume V and tension α as a function of pressure β.
—— exact numerical solution, – – – approximate solutions.

for small β and (40) for large β, are extremely good approximates. The results are shown in Figs. 3 and 4. We see that our asymptotic solutions, not only valid for large β, can often be extended down to small β as well. This is because the coefficients for higher-order corrections in (40), (43), (44) are progressively smaller. As a consequence, our approximate solutions adequately cover the whole range of β except perhaps for the interval 0.2 to 0.4.

Fig. 5 shows the shapes of the membrane cylinder for various β using the exact numerical solution.

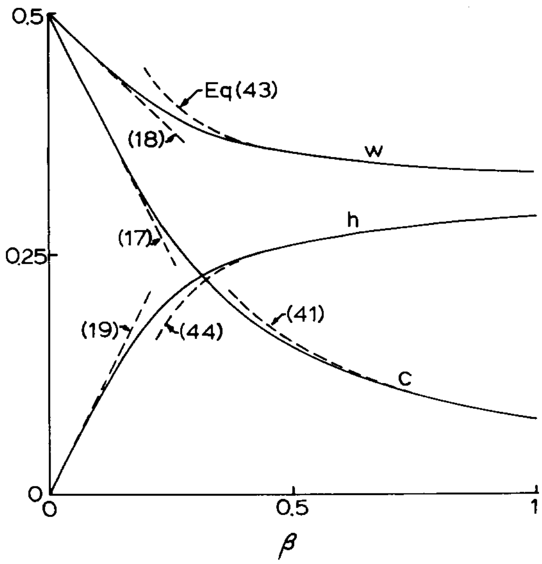

Figure 4. Width w, height h, and contact length c as a function of β. —— exact numerical solution, – – – approximate solutions.

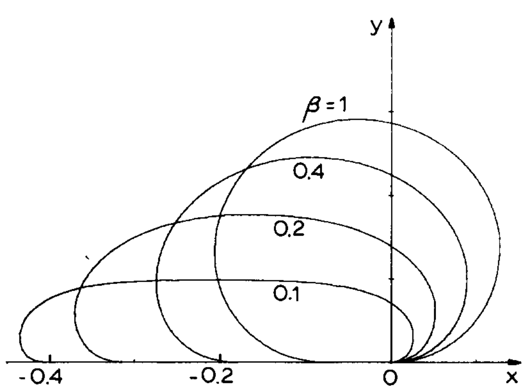

Figure 5. Cross-sectional shapes for various β.

5. Discussion

The tension in the membrane α, a design criterion, increases first quadratically and then linearly with the increase of pump pressure β. Since the volume V already reaches 95% of its maximum at $\beta = 0.45$, it is perhaps not advisible to try to fill the membrane container completely. In fact, the largest change of slope for V is near $\beta = 0.3$ where it is almost 90% filled. The work done per unit width for pumping is

$$\text{work} = \int (P_0 - P_a)\, dV' = \rho g L^3 \int_0^V \beta\, dV. \tag{46}$$

It is proportional to the area between the line $\beta = 0$ and $V(\beta)$ in Fig. 3. The work increases rapidly as β is increased beyond $\beta = 0.3$.

Our results can also be applied to air-filled flexible tubes immersed in water, e.g. pontoons. In that case Fig. 5 should be viewed upside down.

Does Fig. 5 represent the shapes of fluid held together by surface tension? The answer is no. The angle of contact for our problem, $\theta(0) = 0°$, does not correspond to any known fluid-solid system. The lowest angle attainable is mercury on glass for which $\theta(0) = 40°$.

The present work is analogous to a problem in the theory of elastica (see e.g. [3]). Suppose a segment of a thin, circular elastic loop is clamped flat, then the resulting shape is shown in Fig.5. The corresponding parameters are

$$\alpha = \frac{EI}{FL^2}, \quad \beta = \frac{M}{FL} \tag{47}$$

where L is the perimeter length of the loop, EI is the flexural rigidity, M is the moment and F is the force at the origin.

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